

Novel bound states treatment of the two dimensional Schrödinger equation with pseudocentral potentials plus multiparameter noncentral potential

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Abstract By converting the rectangular basis potential $V(x, y)$ into the form as $V(r) + V(r, \varphi)$ described by the pseudo central plus noncentral potential, particular solutions of the two dimensional Schrödinger equation in plane-polar coordinates have been carried out through the analytic approaching technique of the Nikiforov and Uvarov. Both the exact bound state energy spectra and the corresponding bound state wavefunctions of the complete system are determined explicitly and in closed forms. Our presented results are identical to those of the previous works and they may also be useful for investigation and analysis of structural characteristics in a variety of quantum systems.

Keywords Pseudocentral potentials · Noncentral potential · Two dimensional schrödinger equation · Bound states

1 Introduction

The pioneering proposal of Hartmann [1] namely as the ring-shaped potential and its applications [2,3] has offered much attention in recent times. The so-called Hartmann ring-shaped potential $V(r, \varphi)$ exhibits a special case for the noncentral class potentials. It can be expressed as the combinations of an attractive Coulombic term (A/r) and a repulsive noncentral term ($B/r^2 \sin^2 \varphi$) together with the parameters $A = 2a_0 \varepsilon_0 \eta \sigma^2$ and $B = -qa_0 \varepsilon_0 \eta^2 \sigma^2$. Here also $a_0 = \hbar^2/m e^2$, $\varepsilon_0 = -m e^4/2\hbar^2$ stand for the Bohr radius, the ground states energy of the hydrogen atom respectively with the dimensionless parameters q , η and σ varying from 1 up to 10.

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By proving the evidence of ‘accidental’ degeneracies in quantum levels, the dynamical invariance of algebra for the Hartmann potential was firstly handled by the authors Kibler, Negadi [4], Gerry [5] and Kibler, Winternitz [6] respectively. On behalf of the work [6], they have realized that the dynamical symmetry of quantum systems gives rise to the ‘accidental’ degeneracy which is exhibited the discrete spectrum. Zhedanov [7] also proposed the dynamical group symmetry approach to prove the ‘hidden’ symmetry algebra of the Hartmann and oscillator types ring-shaped potential systems.

Two distinct forms of noncentral class potentials are realized in the literature. The first form refers to the Hartmann type ring-shaped potential. The second form firstly introduced by Quesne [8] to investigate the ‘accidental’ degeneracy corresponds to the oscillator type ring-shaped potential. It is employed by combining the terms of harmonic oscillator ($\sim r^2$) and (r, φ) -dependent noncentral potential [9]. It is pointed out that in the first classes the leading term implies the Coulombic discrete energies though in the second ones the leading term suggests the harmonic oscillator basis discrete energies.

Quite a few applications have been encountered in such works on the subject of noncentral potentials (NCPs) from past to the present time. For example, general relations between the elastic constants and the central forces in hexagonal materials as well as noncentral forces in *fcc* monatomic structures were given by Johnson [10, 11]. Some certain structures with these type potentials were also studied by the authors [12–18]. Ermakov type invariants were also discussed with respect to them by Makowski in exact classical limit of quantum mechanics [19]. Besides, the derivations were achieved for determining the stress and elastic constants in systems of particles via noncentral two-body potentials by Murat and Kantor [20]. Several authors has employed them for the aspect of scattering analysis in continuum bound states [21–28]. Fredholm theory was applied to the Lippmann-Schwinger equation and the generalized Levinson theorem was also proved for NCPs [21]. Forward scattering in a system with those potentials was discussed by [23].

Recently, much considerable effort for variety forms of NCPs has been expanded on the solutions of Schrödinger, Dirac and Klein-Gordon equations. The Feynman’s path integral treatment [29–34] and the Green’s function technique [35, 36], the (Lie) algebraic/group theoretical approach [37–41], nonbijective canonical transformation [4, 42], supersymmetric (SUSY) quantum mechanical formalism [43–51] and the NU-analytic method [52–63] as well as the applications for both relativistic [64–77] and other nonrelativistic [78–89] cases are available in the literature.

Due to the above reasons, NCPs may be helpful for determining the structural properties of systems in such cases (e.g. elasticity and stress factors, point-defects of surfaces etc.) in variety fields of physics and chemistry. The ring-shaped like structures, i.e., cyclic polyenes, benzene and benzene-like structures (e.g. graphene), and interactions between deformed pair of nuclei are good examples for the discussion.

The key idea ‘noncentrality concept’ of all potentials plays a privilege role to provide the extensive solutions for concerning quantum systems. That is, if one of NCPs preserves the separability condition for achieving the solutions of system, it can therefore be split into two parts for such cases of radial and angular dependent equations that they may readily be solved.

Our primary goal of this work is to present the expressions of the exact bound states both the energy spectra and the wavefunctions of the two dimensional Schrödinger equation by proposing two definite potential forms in which they are called the pseudo-central potentials plus multiparameter NCP. It is the author's hope to further contribute to applying our approach and results for the most recent quantum systems and for exactly solvable systems with centrally style potentials [90,91]. It may also be applicable for conforming the initial boundary-value problems of certain physical systems (e.g. Dirichlet, Neumann etc.) in two-dimension [92–94].

The organization scheme of the study is given as follows: Section 2 is devoted to construction of the two different rectangular potentials and converting of them to be solvable form for the Schrödinger equation in plane polar coordinates. Section 3 covers the solutions of separated equations via the NU-method. Section 4 is responsible for dealing with the conclusions and remarks.

2 Potential cases and separation of variables

Here we will firstly describe the essential steps of converting the rectangular basis potentials as $V_I(x, y)$ and $V_{II}(x, y)$ into the plane polar forms as $V_I(r) + V_I(r, \varphi)$ and $V_{II}(r) + V_{II}(r, \varphi)$. Secondly, by applying the separational procedure to the two dimensional Schrödinger equation (2-dim SE) then we are going to deal with the solutions of separated equations.

We propose the rectangular form physical potentials as follows

$$V_I(x, y) = \left(\frac{A_0}{\sqrt{x^2 + y^2}} + B_0 \right)^2 + \frac{1}{x^2 y^2} \left[\frac{Bx^4 + Cy^4}{(x^2 + y^2)} + Dx^2 + Fy^2 + G(x^2 + y^2) \right] \quad (1)$$

and

$$V_{II}(x, y) = \left(A_1 \sqrt{x^2 + y^2} + \frac{B_1}{\sqrt{x^2 + y^2}} \right)^2 + \frac{1}{x^2 y^2} \left[\frac{Bx^4 + Cy^4}{(x^2 + y^2)} + Dx^2 + Fy^2 + G(x^2 + y^2) \right] \quad (2)$$

The squared leading terms in Eqs. (1) and (2) refer to the pseudo Coulombic/modified Kratzer and the pseudoharmonic circular oscillator potentials as well. The rest parts of them correspond to the equivalent multiparameter potentials which will be converted into the noncentral case.

Let us consider the cartesian coordinate transformations as [95] $x = r \cos \varphi$, $y = r \sin \varphi$, $\varphi = \tan^{-1}(y/x)$ and $r = \sqrt{x^2 + y^2}$, by putting these into the above equations then they are readily converted into the plane polar forms as

$$V(\mathbf{r}) = V_I(r) + V_{II}(r, \varphi) = \left(\kappa_0 + \frac{\kappa_1}{r} + \frac{\kappa_2}{r^2} \right) + \frac{1}{r^2} \left\{ B \cot^2 \varphi + C \tan^2 \varphi + D \csc^2 \varphi + F \sec^2 \varphi + G \sec^2 \varphi \csc^2 \varphi \right\} \quad (3)$$

with the constants $\kappa_0 = B_0^2 = D_e$, $\kappa_1 = 2A_0B_0 = 2D_e r_e$ and $\kappa_2 = A_0^2 = D_e r_e^2$ as in Ref. [55]. The consecutive terms, in parenthesis, refer to the pseudocentral potential (PCP1) called as the pseudo Coulombic/modified Kratzer case and the rest part constitutes the multiparameter noncentral potential labelled as ($mp - NCP$). By applying the conversion process just only the Eq. (2) gives

$$V(\mathbf{r}) = V_{II}(r) + V_{III}(r, \varphi) = V_0 + A_1^2 r^2 + \frac{B_1^2}{r^2} + (mp - NCP) \quad (4)$$

where we have used the notations $V_0 = 2A_1B_1 = (\kappa/4)r_0^2$ with $A_1 = (\kappa/8)^{1/2}$, and $B_1 = (\kappa/8)^{1/2} r_0^2$ as [91]. The consecutive three-term, in Eq. (4), establishes the pseudocentral harmonic oscillator potential (PCP2).

Let us now regard the two dimensional rectangular basis Schrödinger equation (SE). It can be written

$$\frac{\partial^2 \Psi}{\partial x^2} + \frac{\partial^2 \Psi}{\partial y^2} + \frac{2m}{\hbar^2} [E - V(x, y)] \Psi(x, y) = 0. \quad (5)$$

By employing the task of the polar coordinate transformation on Eq. (5), we rewrite it as

$$\left[\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2}{\partial \varphi^2} + \frac{2m}{\hbar^2} [E - V(\mathbf{r})] \right] \Psi(r, \varphi) = 0. \quad (6)$$

The separation of variables procedure cannot be carried out when rectangular coordinates are employed since the potential energies (1) and (2) are functions of $(x^2 + y^2)^{-1}$ and $(x^2 + y^2)^{-1/2}$. Due to the fact that they are not split into terms, it is required changing to the plane polar coordinates.

Using the ansatz wavefunction in Eq. (6) for successively concerning the Eqs. (3) and (4),

$$\Psi(r, \varphi) = \frac{1}{\sqrt{r}} U(r) \Phi(\varphi) \quad (7)$$

leads to the following equations in which two of them are radial-dependent correspond to the first and second classes of NCPs

$$\frac{d^2 U_A}{dr^2} + \left(\tilde{E}_{HRS} - \frac{\Lambda_0}{r} - \frac{\Lambda}{r^2} \right) U_A(r) = 0, \quad (8)$$

and

$$\frac{d^2U_B}{dr^2} + \left(\tilde{E}_{RSO} - \tilde{A}r^2 - \frac{\Gamma}{r^2} \right) U_B(r) = 0, \tag{9}$$

and the angle-dependent equation

$$\frac{d^2\Phi}{d\varphi^2} + \left\{ M^2 - \left[\left(\frac{\bar{D} + \bar{B} \cos^2 \varphi}{\sin^2 \varphi} \right) + \left(\frac{\bar{F} + \bar{C} \sin^2 \varphi}{\cos^2 \varphi} \right) + \left(\frac{\bar{G}}{\sin^2 \varphi \cos^2 \varphi} \right) \right] \right\} \Phi(\varphi) = 0. \tag{10}$$

In these equations, we use the specifications $\tilde{E}_{HRS} = 2m(E - \kappa_0)/\hbar^2$, $\Lambda_0 = (2m\kappa_1)/\hbar^2$ and $\Lambda = [\bar{\kappa}_2 + (M^2 - \frac{1}{4})]$ with $\bar{\kappa}_2 = (2m\kappa_2)/\hbar^2 = (2mA_0^2)/\hbar^2$; $\tilde{E}_{RSO} = 2m(E - V_0)/\hbar^2$, $\tilde{A} = (2mA_1^2)/\hbar^2$, and $\Gamma = [\bar{B}_1 + (M^2 - \frac{1}{4})]$ with $\bar{B}_1 = (2mB_1^2)/\hbar^2$. In addition, \tilde{E}_{HRS} and \tilde{E}_{RSO} belong to the energies of the Hartmann ring-shaped (*HRS*) and the ring-shaped oscillator (*RSO*) potentials, respectively.

Besides we label the energy parameters for noncentral potential part as $\bar{B} = (2mB)/\hbar^2$, $\bar{C} = (2mC)/\hbar^2$, $\bar{D} = (2mD)/\hbar^2$, $\bar{F} = (2mF)/\hbar^2$ and $\bar{G} = (2mG)/\hbar^2$ with the separation constant M recognized as the angular momentum.

3 Achieving the solutions of separated equations

3.1 Solution of the φ -angle dependent equation

Let us begin firstly to examine some solutions of the equation (10). By introducing a transformation as $t = \sin^2 \varphi$ hence the transformed hypergeometric equation holds a form as

$$\frac{d^2\Phi}{dt^2} + \frac{(1 - 2t)}{2t(1 - t)} \frac{d\Phi}{dt} + \frac{1}{[2t(1 - t)]^2} (\alpha t^2 + \beta t + \gamma) \Phi(t) = 0, \tag{11}$$

where the parameters are $\alpha = (-M^2 + \bar{B} - \bar{C})$, $\beta = (M^2 - \bar{D} - \bar{F})$ and $\gamma = -(\bar{B} + \bar{D} + \bar{G})$. By following [52], a hypergeometric equation is described by

$$\frac{d^2u}{dt^2} + \frac{\tilde{\tau}(t)}{\sigma(t)} \frac{du}{dt} + \frac{\tilde{\sigma}(t)}{\sigma^2(t)} u(t) = 0, \tag{12}$$

then comparing the Eqs. (11) and (12) term-by-term allow us to write the polynomials

$$\tilde{\tau}(t) = (1 - 2t) \quad \sigma(t) = 2t(1 - t) \quad \text{and} \quad \tilde{\sigma}(t) = (\alpha t^2 + \beta t + \gamma). \tag{13}$$

They are all used in solving of the quadratic equation

$$\pi(t) = \frac{1}{2}(\sigma' - \tilde{\tau}) \pm \frac{1}{2}\sqrt{(\sigma' - \tilde{\tau})^2 + 4(k\sigma - \tilde{\sigma})}, \quad (14)$$

where the prime factor of σ denotes the differential at first degree. In this equation the determination of k is very essential step for the calculation of $\pi(t)$ in which the inner part of the square root is required to become the square form with respect to the polynomials.

The substitution process of polynomials for $\pi(t)$ and the arrangement of it for k provide us

$$\pi(t) = \frac{1}{2}(1 - 2t) \pm \frac{1}{2}\sqrt{a_1 t^2 + a_2 t + a_3} \quad (15)$$

with the parameters $a_1 = (-4\alpha - 8k + 4)$, $a_2 = (-4\beta + 8k - 4)$ and $a_3 = (1 - 4\gamma)$. The values of k which bring about the releasing from the square root of $\pi(t)$ can be determined by solving the quadratic equation

$$k^2 + \left(-\beta - 2\gamma - \frac{1}{2}\right)k + \frac{1}{4}(\beta^2 + 2\beta + \alpha + 4\gamma(1 - \alpha)) = 0. \quad (16)$$

Then it yields the double roots as follows

$$k_{1,2} = \left(\gamma + \frac{\beta}{2} + \frac{1}{4}\right) \pm \frac{1}{4}\sqrt{(1 - 4\gamma)\{1 - 4[\gamma + (\alpha + \beta)]\}}. \quad (17)$$

By inserting these roots into the Eq. (15) leads to the fourfold roots for $\pi(t)$ as

$$\pi_{1,2}(t) = \frac{1}{2}(1 - 2t) \pm \frac{1}{4} \left\{ \left[\sqrt{1 - 4\gamma} - \sqrt{1 - 4[\gamma + (\alpha + \beta)]} \right] t - \sqrt{1 - 4\gamma} \right\} \quad (18)$$

for negative root of $k = k_1$ and

$$\pi_{3,4}(t) = \frac{1}{2}(1 - 2t) \pm \frac{1}{4} \left\{ \left[\sqrt{1 - 4\gamma} + \sqrt{1 - 4[\gamma + (\alpha + \beta)]} \right] t - \sqrt{1 - 4\gamma} \right\} \quad (19)$$

for positive root of $k = k_2$, respectively. The resulting Eqs. (13), (17), (18) and (19) play a key role to establish both the energy spectra and the wavefunctions of the system.

3.2 Bound states eigenvalues

Now we can put forward the procedure for determining the energy spectra of the systems. For the purpose of this, we again follow [52] and a novel form of the energy equation is adapted as

$$\begin{aligned} \lambda \equiv \lambda_n &= k + \frac{d\pi}{dt} \\ &= -n \left(\frac{d}{dt}(\tilde{\tau} + 2\pi) - 2n + 2 \right). \end{aligned} \tag{20}$$

Each root k_1 for $\pi_{1,2}(t)$ and k_2 for $\pi_{3,4}(t)$ can be applied which one of them is satisfied the condition $\tau' < 0[\tau' = \frac{d}{dt}(\tilde{\tau} + 2\pi)]$. Possible solutions have often been carried out by the negative roots of the functions $\pi(t)$. Hence we accomplish the result of the angular momentum M which implies the reasonable meaning for physically

$$\begin{aligned} M^2 &= [\bar{D} + \bar{F} + 2(\bar{B} + \bar{G})] + \frac{1}{2} \left(-1 + \sqrt{[1 + 4(\bar{B} + \bar{D} + \bar{G})][1 + 4(\bar{C} + \bar{F} + \bar{G})]} \right) \\ &+ \left\{ (2n_0 + 1) \left[(2n_0 + 1) + \frac{1}{2}(\sqrt{1 + 4(\bar{B} + \bar{D} + \bar{G})} + \sqrt{1 + 4(\bar{C} + \bar{F} + \bar{G})}) \right] + 1 \right\} \end{aligned} \tag{21}$$

with $n_0 = 0, 1, 2 \dots$

One of our basic interests in this step is to show and present how to get the energy eigenvalue results for the radial-dependent Eqs. (8) and (9) with the potential functions (3) and (4) respectively by employing the analogy procedure. It is obvious that the equation (38) and its converted equation (50) given as [53] are identical form to the Eq. (8). Accordingly, it should also admit to be similar form solutions. Following the same procedure as in [53] and accepting the energy parameters of [55] give rise to

$$\begin{aligned} E_{n_1} &= \kappa_0 - \frac{\hbar^2 \Lambda_0^2}{8m} \left[\left(n_1 + \frac{1}{2} \right) + \frac{1}{2} \sqrt{1 + 4\Lambda} \right]^{-2} \\ &= D_e - \frac{2m}{\hbar^2} D_e^2 r_e^2 \left[\left(n_1 + \frac{1}{2} \right) + \frac{1}{2} \sqrt{\frac{2m D_e r_e^2}{\hbar^2} + M^2} \right]^{-2}, \quad n_1 = 0, 1, \dots \end{aligned} \tag{22}$$

Note that the equation (57) in [53] and the energy expression (22) referred to (PCP1) plus ($mp - NCP$) potential exhibit similar characteristic by virtue of the presence of the Coulombic term in potential energy.

The energy spectra expression for (PCP2) plus ($mp - NCP$) can also be determined analogously by dealing with [61] and [91]. The Eq. (15) and its converted form (18) in [61] hold similar structure to the form of the equation given by (9). Following the procedure as [61] step-by-step and the arrangement of the result allow us to write analogously

$$\begin{aligned} E_{n_2} &= V_0 - \hbar\omega \left[(2n_2 + 1) + \frac{1}{2} \sqrt{1 + 4\Gamma} \right] \\ &= \frac{1}{8} \kappa r_0^2 - \frac{\hbar}{2} \sqrt{\frac{\kappa}{m}} \left[(2n_2 + 1) + \sqrt{\left(\frac{m}{4\hbar^2} \right) \kappa r_0^4 + M^2} \right], \quad n_2 = 0, 1, \dots \end{aligned} \tag{23}$$

where $\kappa = 4m\omega^2$ with $\omega = \sqrt{(2mA_1^2)/\hbar^2}$ and the angular momentum M .

3.3 Bound state wavefunctions

The essential steps for performing a set of wavefunction for the complete system should now be examined. In this point we are going to introduce the proper notations in the solutions of $k = k_2 = \gamma + \frac{\beta}{2} + \frac{1}{4}(\sqrt{(1+\beta_1)(1+\beta_2)})$ for which this is the correspondence polynomial of $\pi(t) = \pi_4(t) = \frac{1}{2}(1-2t) - \frac{1}{4}\{[\sqrt{1+\beta_1} + \sqrt{1+\beta_2}]t - \sqrt{1+\beta_1}\}$ with $\beta_1 = -4\gamma$ and $\beta_2 = -4[\gamma + (\alpha + \beta)]$ as well as α , β and γ are given in the Eq. (11).

Let us start to write the equation

$$\frac{\pi(t)}{\sigma(t)} = \frac{d}{dt}[\ln \phi(t)] \quad (24)$$

and from the Eq. (13), straightforward calculations give for

$$\phi(t) = [t(1-t)]^{\delta/2} \quad (25)$$

where $\delta = [1 + (\sqrt{1+\beta_1} + \sqrt{1+\beta_2})/4]$. The weight function [52] $\varrho(t)$ is determined by

$$\frac{d}{dt}(\varrho\sigma) = \varrho\tau \quad (26)$$

with the expression $\tau = \tilde{\tau} + 2\pi$. The calculating procedure yields

$$\varrho(t) = [t(1-t)]^{\delta-1} \quad (27)$$

where $\delta - 1 = (\sqrt{1+\beta_1} + \sqrt{1+\beta_2})/4$. Then the Rodriguez formula is

$$y_n(t) = \frac{C_n}{\varrho(t)} \frac{d^n}{dt^n} [\sigma^n(t)\varrho(t)], \quad (28)$$

where C_n is the normalization constant. Use of $\varrho(t)$ and $\sigma(t)$ provides

$$y_n(t) = \bar{C}_n P_n^{(\mu_1, \mu_2)}(t), \quad t = \sin^2 \varphi \quad (29)$$

with $\bar{C}_n = 2^n C_n$, $\mu_1 = \sqrt{1+\beta_1}$ and $\mu_2 = \sqrt{1+\beta_2}$ as well as $P_n^{(\mu_1, \mu_2)}(t)$ stands for the Jacobi polynomial. The net wave function can thus be written as

$$\begin{aligned} \Phi(t) &= \phi(t) y_n(t) \\ &= \bar{C}_{n_0} [t(1-t)]^{\delta/2} P_{n_0}^{(\mu_1, \mu_2)}(t) \end{aligned} \quad (30)$$

In order to construct the wavefunctions for the radial-dependent Eqs. (8) and (9), we should consider the equations from (58) to (62) as [53] and the equations from (35) to (42) to as [61], respectively. By means of the analogy procedure one gets

$$\begin{aligned}
 U(r) &\equiv U_A(r) = \phi(r) y_n(r) \\
 &= \bar{C}_{n_1} r^p e^{-\sqrt{E_{HRS}} r} L_{n_1}^p(r)
 \end{aligned}
 \tag{31}$$

and

$$\begin{aligned}
 U(s) &\equiv U_B(s) = \phi(s) y_n(s) \\
 &= \bar{C}_{n_2} s^{\zeta/4} e^{-\frac{1}{2}\sqrt{A} s} L_{n_2}^q(s), \quad s = r^2
 \end{aligned}
 \tag{32}$$

where we have used the short notations $p = 1 + 2\sqrt{1 + 4\Lambda}$, $\zeta = 1 + \sqrt{1 + 4\Gamma}$ and $q = (\zeta - 1)/2$. Also $L_{n_1}^p(r)$ and $L_{n_2}^q(s)$ stand for the Laguerre polynomials, respectively.

3.4 Complete solutions of bound wavefunctions

In this part it is required to gather all wavefunctions in which they are satisfied by the equations of (8), (9) and (10) with respect to the ansatz wavefunction (7). If we substitute the Eqs. (30) and (31) into the Eq. (7) one yields

$$\Psi(r, \varphi) = \bar{N} r^{(p-\frac{1}{2})} e^{-\sqrt{E_{HRS}} r} L_{n_1}^p(r) (\sin 2\varphi)^{2\delta} P_{n_0}^{(\mu_1, \mu_2)}(\sin^2 \varphi)
 \tag{33}$$

where $\bar{N} = (\bar{C}_{n_0} \cdot \bar{C}_{n_1})/2^\delta$, $\sin 2\varphi = 2 \sin \varphi \cos \varphi$ and $n_0, n_1 = 0, 1, 2, \dots$. By employing the cartesian coordinate transformations then it is transformed into

$$\begin{aligned}
 \Psi(x, y) &= \bar{N} (x^2 + y^2)^{(p-\frac{1}{2})/2} e^{-\sqrt{E_{HRS}} \sqrt{x^2+y^2}} [xy/(x^2 + y^2)]^{2\delta} \\
 &\quad \times L_{n_1}^p[(x^2 + y^2)^{1/2}] P_{n_0}^{(\mu_1, \mu_2)}[y^2/(x^2 + y^2)].
 \end{aligned}
 \tag{34}$$

In this time we consider the Eqs. (30) and (32) for ansatz Eq. (7) one gets

$$\Psi(r, \varphi) = \tilde{N} r^{(\zeta-1)/2} e^{-\frac{1}{2}\sqrt{A} r^2} L_{n_2}^q(r^2) (\sin 2\varphi)^{2\delta} P_{n_0}^{(\mu_1, \mu_2)}(\sin^2 \varphi)
 \tag{35}$$

with $n_0, n_2 = 0, 1, 2, \dots$. Consequently, it can be converted into

$$\begin{aligned}
 \Psi(x, y) &= \tilde{N} (x^2 + y^2)^{(\zeta-1)/4} e^{-\frac{1}{2}\sqrt{A} (x^2+y^2)} [xy/(x^2 + y^2)]^{2\delta} \\
 &\quad \times L_{n_2}^q(x^2 + y^2) P_{n_0}^{(\mu_1, \mu_2)}[y^2/(x^2 + y^2)],
 \end{aligned}
 \tag{36}$$

where $\tilde{N} = (\bar{C}_{n_0} \cdot \bar{C}_{n_2})/2^\delta$.

4 Conclusions and remarks

In this study we have performed the exact analytical bound state solutions both the energy spectra and the corresponding wavefunctions for the two dimensional Schrödinger equation in plane polar coordinates. The rectangular basis potentials proposed by the Eqs. (1) and (2) are converted into the planar forms in solving of the 2-dim SE through the NU-analytic technique.

Let us remark that the discussion of noncentral class potentials for such quantum systems is limited in the literature. This is the main reason of all not satisfying the condition of analytical solvability. On the other hand, the radial-angular dependency of such potentials will provide us for examining the structural properties of planar quantum systems, i. e. elasticity, stress factor and point-defects of surfaces etc. It is expected that our straightforward approach may therefore offers a solution or provide a model for other identical systems described by [91].

With regard to a variety of potential parameters, possible cases are briefly examined for energy expressions presented by (22) and (23). Certain examples are given as follows:

Letting $B = C = D = F = G = 0$ the angular-dependent Eq. (10) and the Eq. (21) are ignored, therefore this just refers to the energies of the pseudo Coulombic potential (PCP1) and the pseudoharmonic oscillator potential (PCP2). For $D \neq 0$, and $B = C = F = G = 0$, we accomplish the energy eigenvalue results of the pseudo Coulombic type Hartmann ring-shaped potential and the pseudoharmonic oscillator type ring-shaped potential as well.

As a final remark we conclude that the two dimensional angular momentum operator defined as [95]

$$L\Psi = -i\hbar \frac{\partial\Psi}{\partial\varphi}$$

with the eigenvalue ($\hbar M$)

$$L\Psi = (\hbar M)\Psi$$

where the Ψ corresponds to the complete solutions of bound wavefunctions (33) and (35), angular momentum expression of our proposal system is also defined by (21).

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References

1. H. Hartmann, *Theor. Chim. Acta* **24**, 201 (1972)
2. H. Hartmann, R. Schuck, J. Radtke, *Theor. Chim. Acta* **46**, 1 (1976)
3. H. Hartmann, D. Schuck, *Int. J. Quantum Chem.* **18**, 125 (1980)
4. M. Kibler, T. Negadi, *Int. J. Quant. Chem.* **XXVI**, 405 (1984)
5. C.C. Gerry, *Phys. Lett. A* **118**, 445 (1986)
6. M. Kibler, P. Winternitz, *J. Phys. A: Math. Gen.* **20**, 4097 (1987)

7. A.S. Zhedanov, J. Phys. A: Math. Gen. **26**, 4633 (1993)
8. C. Quesne, J. Phys. A: Math. Gen. **21**, 3093 (1988)
9. M.V.C. Bernido, C.C. Bernido, Phys. Lett. A **134**, 395 (1989)
10. R.A. Johnson, Phys. Rev. B **6**, 2094 (1972)
11. R.A. Johnson, J. Phys. F: Metal Phys. **3**, 295 (1973)
12. G.G. Chell, J. Phys. C: Solid State Phys. **1**, 266 (1968)
13. A. Machova, Czech. J. Phys. **24**, 527 (1974)
14. S.K. Mishra, S.S. Kushwaha, IL Nuovo Cimento B **46**, 380 (1978)
15. D.V. Choodnovsky, G.V. Choodnovsky, Lett. Al Nuovo Cimento **22**, 31 (1978)
16. H. Hermann, Phys. Stat. Solidi B **104**, K39 (1981)
17. R. Besson, J. Morillo, Phys. Rev. B **55**, 193 (1997)
18. D. Farkas, C. Vailhe, J. Panova, J. Phase Equilib. **18**, 530 (1997)
19. A.J. Makowski, Phys. Rev. A **68**, 022102 (2003)
20. M. Murat, Y. Kantor, Phys. Rev. E **74**, 031124 (2006)
21. R.G. Newton, J. Math. Phys. **18**, 1348 (1977)
22. R.G. Newton, Phys. Rev. Lett. **62**, 1811 (1989)
23. V. Khare, D.J. Kouri, Chem. Phys. Lett. **80**, 262 (1981)
24. A. Guha, S. Mukherjee, J. Math. Phys. **28**, 840 (1987)
25. A.D. Alhaidari, J. Phys. A: Math. Gen. **38**, 3409 (2005)
26. C.Y. Chen, S.H. Dong, Phys. Lett. A **335**, 374 (2005)
27. S.H. Dong, C.Y. Chen, M.L. Cassou, Int. J. Quant. Chem. **105**, 453 (2005)
28. G.A. Kerimov, J. Phys. A: Math. Gen. **40**, 7297 (2007)
29. I. Sokmen, Phys. Lett. A **115**, 249 (1986)
30. I. Sokmen, Phys. Lett. A **132**, 65 (1988)
31. L. Chetouani, I. Guechi, T.F. Hammann, Phys. Lett. A **125**, 277 (1987)
32. J.M. Cai, A. Inomata, Phys. Lett. A **141**, 315 (1989)
33. B.P. Mandal, Int. J. Mod. Phys. A **15**, 1225 (2000)
34. B. Bentag, L. Chetouani, Czech. J. Phys. **50**, 593 (2000)
35. L. Chetouani, L. Guechi, T.F. Hammann, J. Math. Phys. **33**, 3410 (1992)
36. A.N. Vaidya, H.B. Filho, J. Math. Phys. **31**, 1951 (1990)
37. H.B. Filho, A.N. Vaidya, Phys. Lett. A **145**, 69 (1990)
38. C.Y. Chen, C.L. Liu, D.S. Sun, Phys. Lett. A **305**, 341 (2002)
39. S.H. Dong, G.H. Sun, M.L. Cassou, Phys. Lett. A **328**, 299 (2004)
40. S.H. Dong, G.H. Sun, M.L. Cassou, Phys. Lett. A **340**, 94 (2005)
41. G.A. Kerimov, Phys. Lett. A **358**, 176 (2006)
42. M. Kibler, T. Negadi, Phys. Lett. A **124**, 42 (1987)
43. G.G. Blado, Int. J. Quant. Chem. **58**, 431 (1996)
44. G.G. Blado, Theor. Chim. Acta **94**, 53 (1996)
45. R. Dutt, A. Gangopadhyaya, U.P. Sukhatme, Am. J. Phys. **65**, 400 (1997)
46. B. Gönül, İ. Zorba, Phys. Lett. A **269**, 83 (2000); arXiv: quant-ph/0409085
47. G. Chen, Phys. Scr. **70**, 11 (2004)
48. G. Chen, Chin. Phys. **13**, 144 (2004)
49. S.H. Dong, *Factorization Method in Quantum Mechanics* (Springer, Berlin, 2007)
50. J. Sadeghi, Acta Phys. Pol. A **112**, 23 (2007)
51. J. Sadeghi, M. Rostami, A.R. Hojabri, Int. J. Theor. Phys. **48**, 2961 (2009)
52. A.F. Nikiforov, V.B. Uvarov, *Special Functions of Mathematical Physics* (Birkhauser, Basel, 1988)
53. M. Aktaş, R. Sever, J. Math. Chem. **37**, 139 (2005)
54. F. Yaşuk, C. Berkdemir, A. Berkdemir, J. Phys. A: Math. Gen. **38**, 6579 (2005)
55. Y.F. Chen, T.Q. Dai, Phys. Scr. **75**, 274 (2007)
56. S.M. Ikhdair, R. Sever, Int. J. Theo. Phys. **46**, 2384 (2007)
57. S.M. Ikhdair, Chin. J. Phys. **46**, 291 (2008)
58. S.M. Ikhdair, R. Sever, Cent. Eur. J. Phys. **6**, 685 (2008)
59. Ö. Yeşiltaş, Chin. Phys. Lett. **25**, 1172 (2008)
60. O. Bayrak et al., Int. J. Theo. Phys. **47**, 3005 (2008)
61. M. Aktaş, Int. J. Theor. Phys. **48**, 2154 (2009)
62. C. Berkdemir, J. Math. Chem. **46**, 139 (2009)
63. M.C. Zhang, G.H. Sun, S.H. Dong, Phys. Lett. A **374**, 704 (2010)

64. H.D. Doebner, E. Papp, *Phys. Lett. A* **144**, 423 (1990)
65. Q.W. Chao, *Chin. Phys.* **12**, 136 (2003)
66. Q.W. Chao, *Chin. Phys.* **12**, 1054 (2003)
67. A.D. Alhaidari, *Ann. Phys.* **320**, 453 (2005)
68. C.Y. Chen, *Phys. Lett. A* **339**, 283 (2005)
69. S.H. Dong, M.L. Cassou, *Phys. Scr.* **74**, 285 (2006)
70. C.Y. Chen, D.S. Sun, F.L. Lu, *Phys. Scr.* **74**, 405 (2006)
71. C.Y. Chen, F.L. Lu, D.S. Sun, *Comm. Theo. Phys.* **45**, 889 (2006)
72. Y.F. Cheng, T.Q. Dai, *Comm. Theo. Phys.* **48**, 431 (2007)
73. C.Y. Chen, F.L. Lu, D.S. Sun, *Phys. Scr.* **78**, 015014 (2008)
74. S.M. Ikhdair, R. Sever, *Cent. Eur. J. Phys.* **6**, 141 (2008)
75. M.R. Setare, S. Haidari, *Int. J. Theo. Phys.* **48**, 3249 (2009)
76. M.C. Zhang, *Int. J. Theo. Phys.* **48**, 2625 (2009)
77. X.Y. Jun, L.Z. Wen, *Comm. Theo. Phys.* **53**, 54 (2010)
78. J.J.G. Scanio, *Am. J. Phys.* **45**, 733 (1977)
79. C.F. Souza, M. Gandelman, L.C. Albuquerque, *J. Phys. A: Math. Gen.* **22**, L533 (1989)
80. R.S. Kaushal, *Classical and Quantum Mechanics of Noncentral Potentials* (Springer, Berlin, 1998)
81. R.S. Kaushal, *Ann. Phys.* **205**, 228 (1991)
82. R.S. Kaushal, *Ann. Phys.* **206**, 90 (1991)
83. I.I. Guseinov, *J. Mol. Model.* **9**, 190 (2003)
84. I.I. Guseinov, *J. Math. Chem.* **36**, 83 (2004)
85. H. Taşeli, İ.M. Erhan, Ö. Uğur, *J. Math. Chem.* **32**, 323 (2002)
86. Y. Kasri, L. Chetouani, *Can. J. Phys.* **86**, 1083 (2008)
87. N. Kandırmaz, N. Ünal, *Theo. Math. Phys.* **155**, 884 (2008)
88. A.S. Halberg, E.Z. Gallardo, J.J. Pena, *Int. J. Quant. Chem.* **109**, 1464 (2009)
89. M.C. Zhang, B. An, H.F.G. Qing, *J. Math. Chem.* **48**, 876 (2010)
90. S.H. Dong, *Int. J. Theor. Phys.* **39**, 1119 (2000)
91. S.H. Dong, *Appl. Math. Lett.* **16**, 199 (2003)
92. B.Y. Leis, G.F. Roach, *Proc. R. Soc. Lond. A* **417**, 353 (1988)
93. A.A. Belov, Y.E. Lozik, *Theo. Math. Phys.* **82**, 62 (1990)
94. V.V. Kravchenko, A. Meziani, *J. Math. Anal. Appl.* **377**, 420 (2011)
95. S. Flügge, *Practical Quantum Mechanics*, vol. 1 (Springer, Berlin, 1971)